Hanuman's tail – continued fractions and Ramanujan

V S S Sastry

The author is a math communicator. He has experimented with all types of communication tools available to take maths to people. He creates math cartoons, builds math models, does glove puppetry for math tales, teaches math activities through origami, designs popup books, and writes scripts for math videos. He can be reached at <vsssastry@gmail.com>.

P. C. Mahalanobis, who was a student at Cambridge and who later became an eminent statistician and founder of the Indian Statistical Institute, Kolkata, was a friend of Srinivasa Ramanujan. One day, while visiting Ramanujan, he read about a problem in Strand Magazine. Ramanujan was in the kitchen, cooking. Mahalanobis tried to solve the problem and thought he would ask his good friend about it. So he turned to Ramanujan and said, "Here's a problem for you." "What problem? Tell me," said Ramanujan, still stirring the vegetables. Mahalanobis read out the problem.

A certain street has between 50 and 500 houses in a row, numbered 1, 2, 3, 4, ... consecutively. There is a certain house on the street such that the sum of all the house numbers to the left side of it is equal to the sum of all the house numbers to its right. Find the number of this house.

Ramanujan: Did you find the house number?
Mahalanobis: Yes, I did. If the street had 15 houses, then the required house number is 6, since the sum of all the house numbers to the left side of it is 1 + 2 + 3 + 4 + 5 = 15 which is equal to the sum of all the house numbers to its right 7 + 8. But unfortunately, this doesn't provide the required answer since it is given that the street has between 50 and 500 houses. I got stuck here.

Ramanujan: That means there are many solutions to the stated problem isn't it?
Mahalanobis: Yes, I think so.

Ramanujan immediately said: Take down my solution which is in the form of continued fraction.
Mahalanobis: What? A continued fraction?

Ramanujan: Yes, take it down.

$$6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6}\cdots}}}}\cdots$$

Ramanujan: This continued fraction provides all possible solutions (infinitely many) to your problem if you ignore the condition of 50 to 500 houses.
Mahalanobis: Oh, I see, let me see what is going on?

$\frac{6}{1}$. First convergent fraction giving the initial solution that Mahalanobis found.

$$6-\frac{1}{6}=\frac{36-1}{6}=\frac{35}{6}$$ Second convergent fraction.

$$6-\cfrac{1}{6-\cfrac{1}{6}}=6-\frac{6}{35}=\frac{210-6}{35}=\frac{204}{35}$$

Third convergent fraction.

$$6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6}}}=6-\frac{35}{204}=\frac{1224-35}{204}=\frac{1189}{204}$$

Fourth convergent fraction.

Here 6, 35, 204, 1189, . . . (the numerators of successive convergent fractions) are the required house numbers. Thus after 6, the second possible house number must be 35 since 1 + 2 + 3 + . . . + 34 = 595 = 36 + 37 + . . . + 49. But again this doesn't fit the solution asked for in the Strand magazine problem, since here, there are only 49 houses in the street. Hence we need to consider the third convergent fraction.

The third convergent fraction having the numerator 204 must be the correct solution to the Strand magazine problem since 1 + 2 + 3 + . . . + 203 = 20706 = 205 + 206 + . . . + 288. This solution suggests that the street has a total of 288 houses (satisfying the condition between 50 and 500 houses) in which the desired house number must be 204.

Note that if we do not restrict ourselves to the condition that the street has between 50 and 500 houses, then we get an infinite number of solutions, thanks to the continued fraction provided by Ramanujan.

That was Ramanujan. Quick in answering. And he used

continued fractions with ease. Look at the dots in the above solution. It means fractions continue to be written endlessly. Hence the name continued fraction. If an analogy can be given, it continues like the tail of Hanuman. Ramanujan was fascinated by this Hamuman's tail – continued fractions. His notebooks reveal that nearly 20% of his jottings are about this.

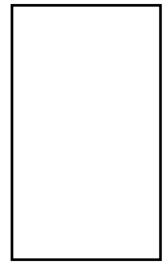
Why was he so attracted to this type of writing? We do not know. But it looks beautiful. It is compact (as in the example above), but gives out infinite solutions.

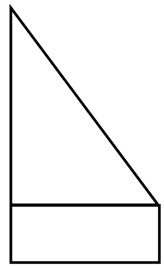
But this beauty, compactness and infinite answers in its womb (continued fractions) is denied to our students. Because continued fractions as a form are dropped from the textbooks. Our teachers and students know or are exposed to only the math of the textbook. That's why they cannot appreciate this episode that illustrates Ramanujan's brilliance. What a tragedy!

We need not go in search of problems for continued fractions. They are everywhere. Even in an ordinary A4 size paper.

Again the story goes like this

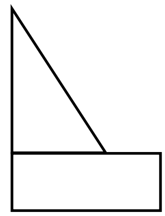
Take an A4 size paper P-1

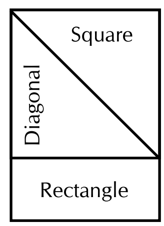




Fold a right angle triangle P-2

Fold the remaining paper up P-3





Open the paper and it looks like this P-4

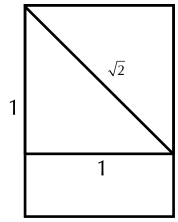
There is a diagonal inside a square. A square has equal sides. We can consider the side to have value 1. Then diagonal = $\sqrt{1^2+1^2}=\sqrt{2}$

(Pythagoras theorem)

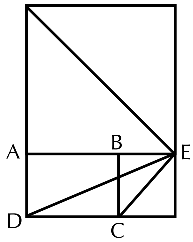
This is exactly the breadth of the A4 size paper. The length is the side of the square. That means the aspect ratio $L{:}B = 1{:}\sqrt{2}$. That's why an A4 size paper is also called $\sqrt{2}$ paper. Now-a-days, it is called a silver rectangle (compared with the golden rectangle 1: 1.618).

Now look at our silver rectangle. We have these measures

Remainder rectangle P-5



Inside the remainder rectangle you can fold a square and its diagonal.



Observe ΔCED is an isosceles triangle (CE = CD). That means excluding the square the remaining rectangle (ABCD) is also a silver rectangle. (*Note: This may be verified by taking AE as 1 unit and working out the other relevant sides using the Pythagoras theorem.*)

As earlier, we can fold a square in this rectangle and as earlier a residual rectangle exists. You can continue this process again and again. Every time a residual rectangle remains.

Now let us go back to the A4 rectangle.

Here L x B = 1X $\sqrt{2}$ = square + remaining rectangle
= 1X1 + remainder

$\sqrt{2}$ = 1 + remainder

As silver rectangles repeat, their aspect ratios (L:B) are maintained. They are similar rectangles. Hence the ratio of length (= 1 unit) and breadth (= 2sq + remainder) is maintained for all. Rectangles get smaller and smaller but retain the aspect ratio to the mother rectangle.

So we get an equation

$$\text{Remainder} = \frac{1}{2 + \text{remainder}}$$

Substitute in previous formula

$$\sqrt{2} = 1 + \frac{1}{2 + \text{remainder}}$$

$$= 1 + \frac{1}{2 + \frac{1}{2 + \text{remainder}}}$$

$$= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \text{remainder} \ldots}}}$$

This is continued fraction for $\sqrt{2}$. Let us see what we get when we consider convergent fractions at every level

$$1\frac{3}{2} = 1.5, \frac{7}{5} = 1.4, \frac{17}{12} = 1.4166$$

$$\frac{41}{29} = 1.4137, \frac{99}{70} = 1.4142\ldots$$

The last one – $\sqrt{2}$ = 1.4142...is recognizable.

This is the beauty of continued fractions.

TP