Figurate numbers

What is a figurate number?

Sequences generated by figures made up of equally spaced dots are known as *figurate* numbers. They include the triangular numbers, square numbers, pentagonal numbers, etc. They are also called *polygonal* numbers as they form polygons when arranged suitably.

Figurate numbers

God made the integers, all else is the work of man. – Leopold Kronecker

Pythagoras (569-475 B.C.)

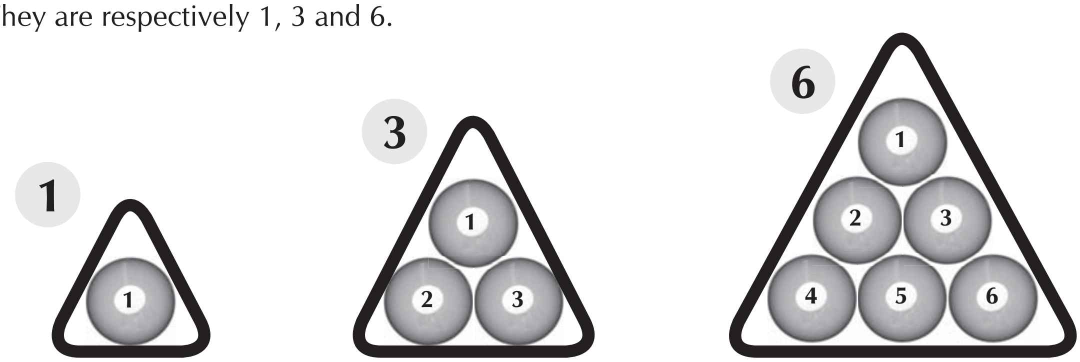
Do you want to learn about a series of numbers that kept mathematicians – both ancient and modern – busy? Yes, figurate numbers have fascinated amateur and professional mathematicians, teachers and children for ages. Let us explore these numbers and their properties.

Pythagoras and his pupils found these numbers attractive and did a special study.

Pascal, the famous French mathematician took special interest in these numbers and wrote a treatise on triangular numbers.

Activity 1

The simplest of the family of figurate numbers are the triangular numbers. Take a set of identical coins or buttons and arrange them in the form of a triangle. Note the number of coins required to make the triangles. They are respectively 1, 3 and 6.



Can you continue, construct and write the next few triangular numbers?

4th triangular number

5th triangular number

6th triangular number

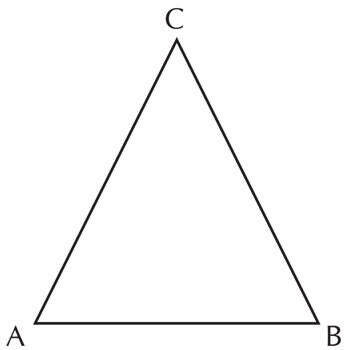
7th triangular number

Activity 2

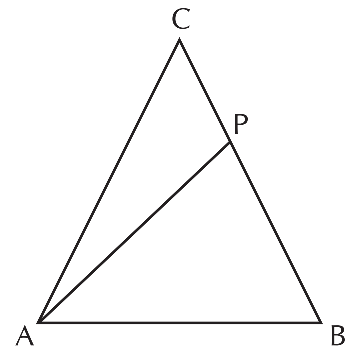
Curiously, triangular numbers appear in unexpected places. Like in this example.

Consider a triangle.

This figure contains 1 triangle

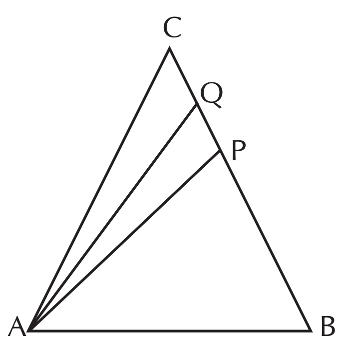


This figure contains 3 triangles



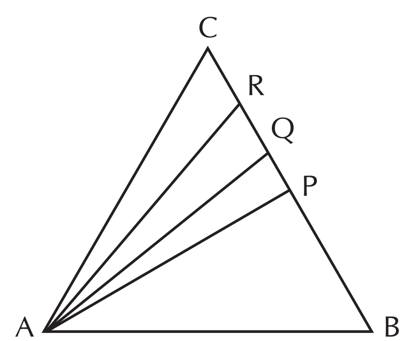
How? Observe carefully.
The triangles are (1) ABC (2) APC (3) APB

The next figure contains six triangles.



They are (1) ABC (2) ABQ (3) ACP (4) ABP (5) APQ (6) AQC

This triangle has 10 hidden triangles. Can you find all 10?



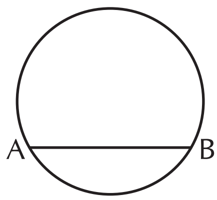
List them here _____ _____ _____ _____

_____ _____ _____ _____ _____ _____

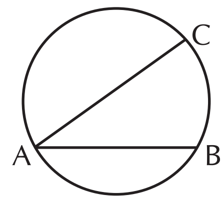
Continue this exercise and determine the number of triangles in the new triangles.

Activity 3

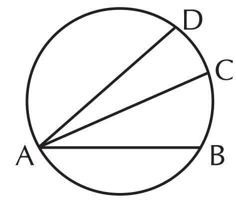
Something similar happens when you consider segments in a circle generated by drawing chords.



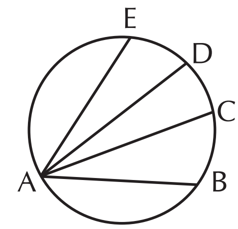
Three segments



Six segments



_____ segments



_____ segments

If the circle activity has not surprised you, the next one will!

Activity 4

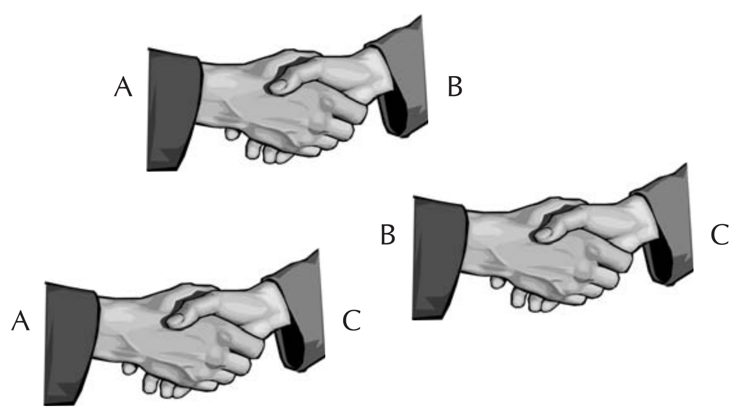
Problem

Six business people meet for lunch and shake hands with each other. How many handshakes are there?

Solution

If two people shake hands there is one handshake.

If three people A, B and C shake hands there are 3 handshakes.



If four people shake hands, there are 3 more handshakes. So $3 + 3 = 6$ in total.

If five people shake hands, there are another 4 handshakes. So $6 + 4 = 10$.

For 6 people there are another 5 handshakes. So $10 + 5 = 15$.

Lo and behold! These are the first five triangular numbers.

Now can you generate the first few triangular numbers?

Activity 5

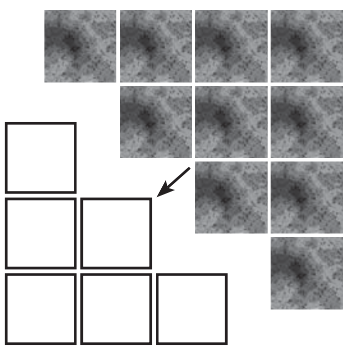
Triangular numbers also exhibit some interesting properties. The sum of two consecutive triangular numbers is *always* a square number.

Eg. $1+3=4$; $3+6=9$; $6+10=16$, etc. Can you verify the above statement by choosing different numbers?

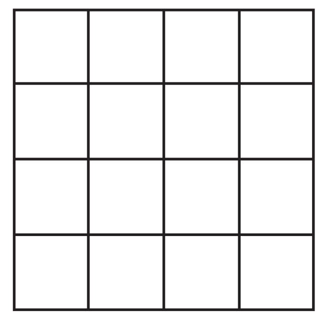
For instance, if we represent nth triangular numbers as T_n then,

$T_n + T_{n+1} = n^2$

This statement has a beautiful graphical proof.

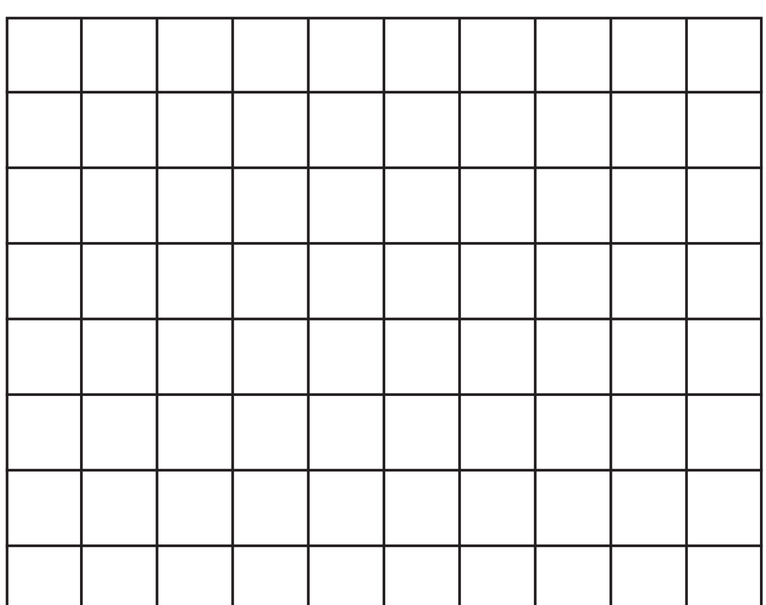


Result



Can you prove the following statement using the square paper?

4^{th} triangular number + 5^{th} triangular number = 4^2 (Hint: Cut out and rejoin)

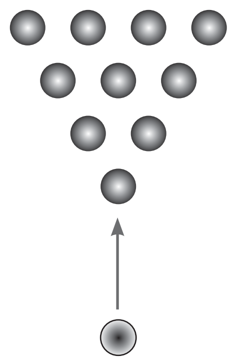


Photocopiable

Activity 6

If you are a keen observer, you will notice triangular numbers around you. For instance,

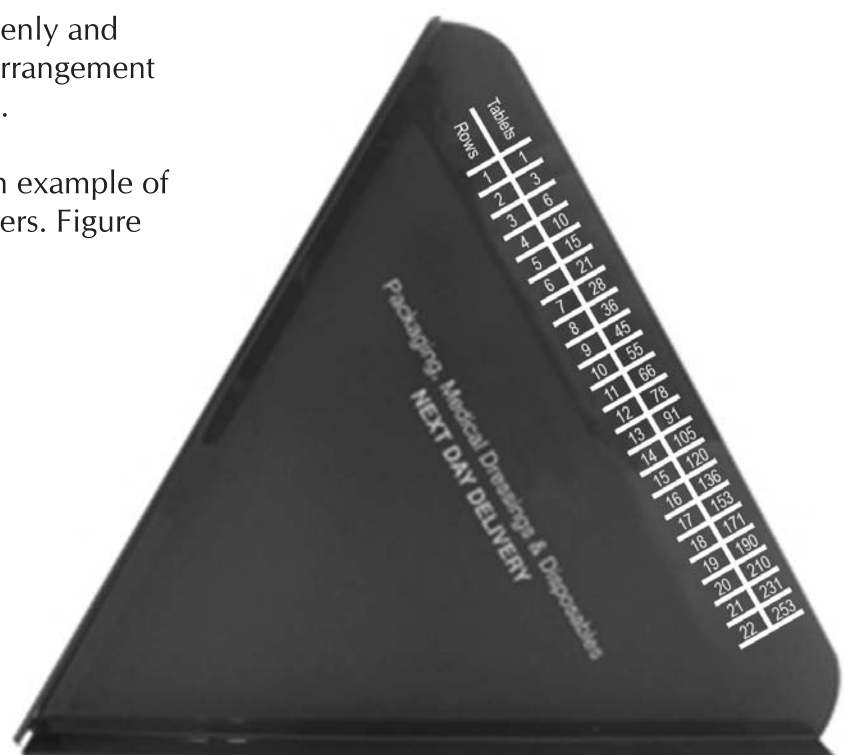
Or in the bowling alley



Have you observed the opening arrangement of the balls in a snooker game? Yes, it is also another example of triangular numbers in real life.

Observe a group photograph keenly and you are bound to notice some arrangement representing triangular numbers.

There are tablet dispensers as an example of practical use of triangular numbers. Figure out how it works.



Activity 7

Properties of triangular numbers

Choose any natural number. Express it as a sum of triangular numbers (repetitions are allowed). Can you *always* do it with a maximum of three triangular numbers? Let us see.

5 = 3 + 1 + 1

9 = 6 + 3

13 = 10 + 3

19 = 15 + 3 + 1 or 10 + 6 + 3

40 = ☐ + ☐ + ☐

31 = ☐ + ☐ + ☐

23 = ☐ + ☐ + ☐

8 = ☐ + ☐ + ☐

68 = ☐ + ☐ + ☐

95 = ☐ + ☐ + ☐

44 = ☐ + ☐ + ☐

50 = ☐ + ☐ + ☐

75 = ☐ + ☐ + ☐

100 = ☐ + ☐ + ☐

130 = ☐ + ☐ + ☐

90 = ☐ + ☐ + ☐

22 = ☐ + ☐ + ☐

23 = ☐ + ☐ + ☐

24 = ☐ + ☐ + ☐

25 = ☐ + ☐ + ☐

26 = ☐ + ☐ + ☐

27 = ☐ + ☐ + ☐

Conclusion: *Every natural number* can be written as a sum of at the most three triangular numbers.

The story of Gauss

There is a well known story about Karl Friedrich Gauss (1777-1855) when he was in elementary school. His teacher got mad at the class and told them to add the numbers 1 to 100 and give him the answer by the end of the class. About 30 seconds later Gauss gave him the answer.

The other kids were adding the numbers like this:
1 + 2 + 3 + ... + 99 + 100 = ?

But Gauss rearranged the numbers to add them like this:
(1 + 100) + (2 + 99) + (3 + 98) + . . . + (50 + 51) = ?

If you notice every pair of numbers adds up to 101. There are 50 pairs of numbers, so the answer is 50x101 = 5050. Of course, Gauss came up with the answer about 20 times faster than the other kids.

In general to find the sum of all the numbers from 1 to n:
1 + 2 + 3 + 4 + ... + n = (1 + n)x(n/2)

As you already know this is the n^{th} triangular number.

That is "1 plus n times n divided by 2."

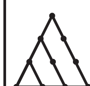
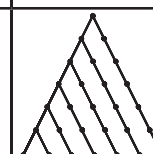
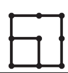
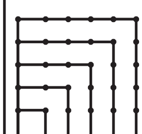
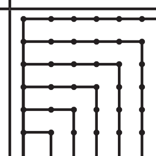
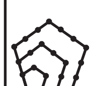
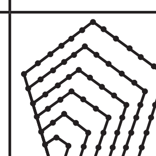
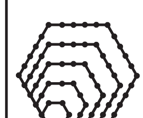
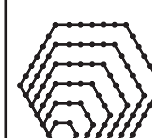
Activity 8

It is now time to move on to other figurate numbers.

Square numbers: Sequence of numbers generated by dots making squares of increasing size.

Pentagonal numbers: Sequence of numbers generated by dots making regular pentagons of increasing size.

The table below gives a list of different figurate numbers. Examine it carefully and fill in the missing picture and data.

Type	1st	2nd	3rd	4th	5th	6th	7th
Traiangular							
Value	1	3	6	10		21	28
Square							
Value	1	4				36	
Pentagonal							
Value	1	5	12				70
Hexagonal							
Value	1	6	15		45		

Why study figurate (polygonal) numbers?

- To understand and explore number patterns and relationships
- To appreciate beauty in maths
- To explore practical uses of these numbers.

Extension:

You may like to explore and understand figurate numbers in a three dimensional space. Try it! Share your experience with TEACHER PLUS.

Developed by Dr. S N Gananath. He can be reached at <sngananath@gmail.com>.